Appendix for *A general mathematical framework for representing soil organic matter dynamics*

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B Theorems and Definitions

Theorems and definitions presented here are used to support the arguments in the manuscript. They were extracted almost literally from the original sources: Braun (1993); Holmes and Shea-Brown (2006); Sontag (2008); d'Andréa Novel and De Lara (2013).

B.1 Definitions

We mainly consider autonomous ordinary differential equations (ODEs), written in vector notation as:

$$\frac{d}{dt}\mathbf{x}=\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x});\ \mathbf{x} \in \mathbb{R}^n, \tag{B.1}$$

where

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix},\ \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{B.2}$$

We denote a solution to (B.1) by $\mathbf{x}(t)$, with initial conditions $\mathbf{x}(0)$. Equilibria $\mathbf{x}^e$ (sometimes called "equilibrium points" or "fixed points"), are special constant solutions: $\mathbf{x}(t) \equiv \mathbf{x}^e$ where $\dot{\mathbf{x^e}} = \mathbf{f}(\mathbf{x}^e) = \mathbf{0}$, which is equivalent to requiring $f_j(x_1^e, \ldots, x_n^e) = 0$ for all $1 \leq j \leq n$.

Definition: 1. Lyapunov stability $\mathbf{x}^e$ *is a 'stable' equilibrium if for every neighborhood* U *of* $\mathbf{x}^e$ *there is a neighborhood* $V \subseteq U$ *of* $\mathbf{x}^e$ *such that every solution* $\mathbf{x}(t)$ *starting in* V $(\mathbf{x}(0) \in V)$ *remains in* U *for all* $t \geq 0$*. Notice that* $\mathbf{x}(t)$ *need not approach* $\mathbf{x}^e$*. If* $\mathbf{x}^e$ *is not stable, it is 'unstable'.*

Definition: 2. Asymptotic stability *An equilibrium* $\mathbf{x}^e$ *is 'asymptotically stable' if it is Lyapunov stable and additionally* V *can be chosen so that* $|\mathbf{x}(t) - \mathbf{x}^e| \to 0$ *as* $t \to \infty$ *for all* $\mathbf{x}(0) \in V$*.*

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Definition: 3. BIBS and BIBO stability *The linear dynamical system*

$$\frac{d\boldsymbol{C}(t)}{dt} = \boldsymbol{I}(t) + \mathbf{T}(\boldsymbol{C}, t) \cdot \mathbf{N}(\boldsymbol{C}, t) \cdot \boldsymbol{C}(t)$$
$$\boldsymbol{r}(t) = \mathbf{R}(t) \cdot \mathbf{N}(\boldsymbol{C}, t) \cdot \boldsymbol{C}(t)$$

is said to be BIBS-stable if, for all initial conditions $\boldsymbol{C}_0$ *and for all bounded inputs* $(\boldsymbol{I}(t), t \geq 0)$*, the state* $(\boldsymbol{C}(t), t \geq 0)$ *remains bounded:*

$$\sup_{t \geq 0} \|\boldsymbol{I}(t)\| < +\infty \Rightarrow \sup_{t \geq 0} \|\boldsymbol{C}(t)\| < +\infty. \tag{B.3}$$

Similarly, the system is said to be BIBO-stable if, for all initial conditions and for all bounded input $(\boldsymbol{I}(t), t \geq 0)$*, the output* $(\boldsymbol{r}(t), t \geq 0)$ *remains bounded:*

$$\sup_{t \geq 0} \|\boldsymbol{I}(t)\| < +\infty \Rightarrow \sup_{t \geq 0} \|\boldsymbol{r}(t)\| < +\infty. \tag{B.4}$$

Definition: 4. ISS *Consider the system:*

$$\dot{\mathbf{X}} = \mathbf{g}(\mathbf{X}, \mathbf{u}) \tag{B.5}$$

Then (B.5) *is said to be locally input-to-state-stable (ISS) if there exist a* $\mathcal{KL}$ *function* β*, a class* $\mathcal{K}_\infty$ *function* γ *and constants* $k_1, k_2 \in \mathbb{R}^+$ *such that*

$$\|\mathbf{X}(t)\| \leq \beta(\|\mathbf{X}_0\|, t) + \gamma(\|\mathbf{u}\|_\infty), \quad \forall t \geq 0 \tag{B.6}$$

for all $\mathbf{X}_0 \in D$ *and* $\mathbf{u} \in D_u$ *satisfying:* $\|\mathbf{X}_0\| < k_1$ *and* $\|\mathbf{u}\|_\infty = \sup_{s>0} \|\mathbf{u}(s)\| < k_2$*. It is said to be input-to-state-stable, or globally ISS if* $D = \mathbb{R}^n, D_u = \mathbb{R}^m$ *and* (B.6) *is satisfied for any initial state and any bounded input* u*.*

Definition: 5. Class $\mathcal{K}_\infty$ function *A class* $\mathcal{K}_\infty$ *function is a function* $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ *which is continuous, strictly increasing, unbounded, and satisfies* $\alpha(0) = 0$.

Definition: 6. Class $\mathcal{KL}$ function *A class* $\mathcal{KL}$ *function is a function* $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\beta(\cdot, t) \in \mathcal{K}_\infty$ *for each* t *and* $\beta(r, t) \searrow 0$ *as* $t \to \infty$*.*

Definition: 7. $\mathbb{R}^{n+}, \mathbb{R}^{n-}$
We call the positive part of the phase space $\mathbb{R}^{n+}$*:*

$$\mathbb{R}^{n+} = \{\mathbf{v} \ : v_i \geq 0 \quad \forall i \in \{1, \dots n\}\}$$

and accordingly

$$\mathbb{R}^{n-} = \{\mathbf{v} \ : v_i \leq 0 \quad \forall i \in \{1, \dots n\}\}$$

Remark:
Neither $\mathbb{R}^{n+}$ *nor* $\mathbb{R}^{n+}$ *nor* $\mathbb{R}^{n+} \cup \mathbb{R}^{n-}$ *are vector spaces. They are just subsets of* $\mathbb{R}^n$ *not sub spaces and the smallest subspace of* $\mathbb{R}^n$ *that contains them is* $\mathbb{R}^n$ *itself.*

B.2 Stability of linear systems

Consider the linear system

$$\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x} \tag{B.7}$$

Theorem 1. *(a) Every solution* $\mathbf{x} = \phi(t)$ *of* (B.7) *is stable if all eigenvalues of* $\mathbf{A}$ *have negative real part.*
(b) Every solution $\mathbf{x} = \phi(t)$ *of* (B.7) *is unstable if at least one eigenvalue of* $\mathbf{A}$ *has positive real part.*
(c) Suppose that all eigenvalues of $\mathbf{A}$ *have real part* ≤ 0 *and* $\lambda_1 = i\sigma_1 \ldots, \lambda_l = i\sigma_l$ *have zero real part. Let* $\lambda_j = i\sigma_j$ *have multiplicity* k_j*. This means that the characteristic polynomial of* $\mathbf{A}$ *can be factored into the form*

$$p(\lambda) = (\lambda - i\sigma_1)_1^k \ldots (\lambda - i\sigma_l)_l^k q(\lambda)$$

where all the roots of $q(\lambda)$ *have negative real part. Then, every solution* $\mathbf{x} = \phi(t)$ *of* (B.7) *is stable if* $\mathbf{A}$ *has* k_j *linearly independent eigenvectors for each eigenvalue* $\lambda_j = i\sigma_j$*. Otherwise, every solution of* $\phi(t)$ *is unstable.*

A proof of this theorem is provided in Braun (1993).

B.3 Hartman-Grobman theorem

The Hartman-Grobman theorem shows that near a hyperbolic equilibrium point $\mathbf{x_0}$, i.e. $Re(\lambda_i) \neq 0$ for $i = 1, \ldots, n$, the nonlinear system of equation (B.1) has the same qualitative structure as the linear system of equation (B.7) with $\mathbf{A} = D\mathbf{f}(\mathbf{x_0})$, where $D\mathbf{f} = [\partial f_i / \partial x_j]$ is the Jacobian matrix of first partial derivatives of $\mathbf{f}$ (Guckenheimer and Holmes, 1983).

Theorem 2. *If* $D\mathbf{f}(\mathbf{x_0})$ *has no zero or purely imaginary eigenvalues then there is a homeomorphism* h *defined on some neighborhood* U *of* $\mathbf{x_0}$ *in* $\mathbb{R}^n$ *locally taking orbits of the nonlinear flow* $\phi(t)$ *of* (B.1)*, to those of the linear flow* $e^{tD\mathbf{f}(\mathbf{x_0})}$ *of* (B.7)*. The homeomorphism preserves the sense of orbits and can also be chosen to preserve parameterization by time.*

Literature cited

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