Appendix B: Obtaining an expression for average fitness under density dependent population growth.

We wish to obtain a simpler expression for (Eq. 19 in the main text):

$$F(\mathbf{f},N_t)=\overbrace{\frac{\int_E F_{\mathbf{x}}w_{\mathbf{x}}f_{\mathbf{x}}\,d\mathbf{x}}{\int_E w_{\mathbf{x}}f_{\mathbf{x}}\,d\mathbf{x}}}^{\substack{\text{Density}\\ \text{independent}\\ \text{component}}}\overbrace{-\,bN_tA^{-1}\frac{\int_E w_{\mathbf{x}}^2f_{\mathbf{x}}\,d\mathbf{x}}{\left(\int_E w_{\mathbf{x}}f_{\mathbf{x}}\,d\mathbf{x}\right)^2}}^{\substack{\text{Density}\\ \text{dependent}\\ \text{component}}} \qquad \text{B.1}$$

The first half of this expression is the density-independent component provided by Supplement I. We focus on the numerator of the density dependent component.

$$\int_E w_{\mathbf{x}}^2f_{\mathbf{x}}\,d\mathbf{x}=\frac{1}{g}\int_E\sum_{l=1}^{L}\psi_l\exp\left(2G_{\mathbf{x}}-\frac{1}{2}\sum_{k=1}^{K}\left(\frac{x_k-\mu_{l,k}}{\sigma_k}\right)^2\right)d\mathbf{x} \qquad \text{B.2}$$

Where $g=\left(2\pi\right)^{\frac{K}{2}}\prod_{k=1}^{K}\sigma_k$ (see eq. 11). Using similar arguments to those used for the denominator in Supplement I, this has the form:

$$\frac{1}{g}\sum_{l=1}^{L}\psi_l\prod_{j=1}^{K}\left(\frac{2\pi\sigma_j^2}{1-4\gamma_{2,j}\sigma_j^2}\right)^{\frac{1}{2}}\exp\left(2\gamma_{0,j}-\frac{\mu_{l,j}^2}{2\sigma_j^2}+\frac{\left(2\gamma_{1,j}\sigma_j^2+\mu_{l,j}\right)^2}{2\sigma_j^2(1-4\gamma_{2,j}\sigma_j^2)}\right) \qquad \text{B.3}$$

We now return to the denominator of the density dependent component of eq. B.1 whose integral is provided by eq. **Error! Reference source not found.**

$$\left(\int_E w_{\mathbf{x}}f_{\mathbf{x}}\,d\mathbf{x}\right)^2=\frac{1}{g^2}\left[\sum_{l=1}^{L}\psi_l\prod_{j=1}^{K}\left(\frac{2\pi\sigma_j^2}{1-2\gamma_{2,j}\sigma_j^2}\right)^{\frac{1}{2}}\exp\left(\gamma_{0,j}-\frac{\mu_{l,j}^2}{2\sigma_j^2}+\frac{\left(\gamma_{1,j}\sigma_j^2+\mu_{l,j}\right)^2}{2\sigma_j^2(1-2\gamma_{2,j}\sigma_j^2)}\right)\right]^2 \qquad \text{B.4}$$

This expression may be re-written as

$$\prod_{j=1}^{K}\left(\frac{2\pi\sigma_j^2}{1-2\gamma_{2,j}\sigma_j^2}\right)\exp\left(\gamma_{0,j}+\frac{\gamma_{1,j}^2\sigma_j^2}{2(1-2\gamma_{2,j}\sigma_j^2)}\right)^2\left[\sum_{l=1}^{L}\psi_l\prod_{j=1}^{K}\exp\left(\frac{2\mu_{l,j}(\mu_{l,j}\gamma_{2,j}+\gamma_{1,j})}{(1-2\gamma_{2,j}\sigma_j^2)}\right)\right]^2 = \qquad \text{B.5}$$

$$\prod_{j=1}^{K}\left(\frac{2\pi\sigma_j^2}{1-2\gamma_{2,j}\sigma_j^2}\right)\exp\left(\gamma_{0,j}+\frac{\gamma_{1,j}^2\sigma_j^2}{2(1-2\gamma_{2,j}\sigma_j^2)}\right)^2\left[\sum_{l=1}^{L}\sum_{m=1}^{L}\psi_l\psi_m\prod_{j=1}^{K}\exp\left(2\frac{\mu_{l,j}(\mu_{l,j}\gamma_{2,j}+\gamma_{1,j})+\mu_{m,j}(\mu_{m,j}\gamma_{2,j}+\gamma_{1,j})}{(1-2\gamma_{2,j}\sigma_j^2)}\right)\right]$$

However, putting together the numerator with this version of the denominator gives limited scope for simplification. We therefore collect all of these results together into an expression for density dependence:

$$F(N_t)=\frac{F_1}{F_2}-bN_t gA^{-1}\frac{F_3}{F_2^2} \qquad \text{B.6}$$

Where

$$F_1=\sum_{l=1}^{L}\psi_l\Theta_l\sum_{j=1}^{K}\left\{\beta_{0,j}+\beta_{1,j}\frac{\left(\gamma_{1,j}\sigma_j^2+\mu_{l,j}\right)}{\left(1-2\gamma_{2,j}\sigma_j^2\right)}+\beta_{2,j}\frac{\sigma_j^2}{\left(1-2\gamma_{2,j}\sigma_j^2\right)}\left(1+\frac{\left(\gamma_{1,j}\sigma_j^2+\mu_{l,j}\right)^2}{\sigma_j^2\left(1-2\gamma_{2,j}\sigma_j^2\right)}\right)\right\}$$

$$F_2=\sum_{l=1}^{L}\psi_l\Theta_l$$

$$F_3=\sum_{l=1}^{L}\psi_l\prod_{j=1}^{K}\left(\frac{2\pi\sigma_j^2}{1-4\gamma_{2,j}\sigma_j^2}\right)^{\frac{1}{2}}\exp\left(2\gamma_{0,j}-\frac{\mu_{l,j}^2}{2\sigma_j^2}+\frac{\left(2\gamma_{1,j}\sigma_j^2+\mu_{l,j}\right)^2}{2\sigma_j^2(1-4\gamma_{2,j}\sigma_j^2)}\right) \qquad \text{B.7}$$

$$\Theta_l=\prod_{j=1}^{K}\left(\frac{2\pi\sigma_j^2}{1-2\gamma_{2,j}\sigma_j^2}\right)^{\frac{1}{2}}\exp\left(\gamma_{0,j}-\frac{\mu_{l,j}^2}{2\sigma_j^2}+\frac{\left(\gamma_{1,j}\sigma_j^2+\mu_{l,j}\right)^2}{2\sigma_j^2(1-2\gamma_{2,j}\sigma_j^2)}\right)$$

$$g=\left(2\pi\right)^{\frac{K}{2}}\prod_{k=1}^{K}\sigma_k$$