Appendix C: Solutions and stability of the facilitation model

Eqs. 25 and 26–28 are set to zero to solve for the equilibria. x_ϕ^* is the same as in the null model and is equal to $h/(r+h)$. We then solve for the equilibrium of the next highest state, x_0^*. The roots of the quadratic are

$$x_0 = \frac{r}{r+h} \text{ and } \frac{h}{c_1}. \qquad \textbf{(C.1)}$$

$x_0^* = h/c_1$ when $x_1 > 0$. Because when $x_0 = r/(r+h)$, $x_\phi + x_0 = 1$, only the second solution makes sense if there are patches in other states.

We repeat the same procedure to solve for x_1, using $x_0 = h/c_1$; the roots are

$$x_1^* = \frac{r}{r+h} - \frac{h}{c_1} \text{ and } \frac{h}{c_2}. \qquad \textbf{(C.2)}$$

Again, the first solution corresponds to a case in which species 2 (and above) are absent. Because of this structure, we can solve for an arbitrary i.

$$\frac{dx_i}{dt} = 0 = -c_{i+1}x_i\left(\sum_{j=i+1}^{S} x_j\right) + c_i x_{i-1}\left(\sum_{j=1}^{S} x_j\right) - hx_i \qquad \textbf{(C.3)}$$

$$0 = -c_{i+1}x_i\left(1 - x_\phi - \sum_{j=0}^{i} x_j\right) + c_i x_{i-1}\left(1 - x_\phi - \sum_{j=0}^{i-1} x_j\right) - hx_i \qquad \textbf{(C.4)}$$

$$0 = -c_{i+1}x_i\left[\frac{r}{r+h} - h\left(\sum_{j=1}^{S}\frac{1}{c_j}\right) - x_i\right] + c_i\frac{h}{c_i}\left[\frac{r}{r+h} - h\left(\sum_{j=1}^{S}\frac{1}{c_j}\right)\right] - hx_i \qquad \textbf{(C.5)}$$

x_i^* is either

$$x_i = \frac{r}{r+h} - h\sum_{j=1}^{S}\frac{1}{c_j} \qquad \textbf{(C.6)}$$

$$x_i = \frac{h}{c_{i+1}}. \qquad \textbf{(C.7)}$$

If $x_i = \dfrac{h}{c_i}$ pushes the sum of all uninhabitable and lower hierarchy patches higher than one, the first solution must hold. Alternatively $\dfrac{r}{r+h} - h\sum_{j=1}^{S}\dfrac{1}{c_j}$ must always be non-negative, since this expresses the proportion of patches 'left over' for state i. When $c_i = c$, this simplifies to $c \geq i\dfrac{h}{r}(r+h)$. The highest species number that can persist, S^* is

$$S^* = \min\left(S, \left\lfloor \frac{cr}{h(r+h)} \right\rfloor\right) \qquad \textbf{(C.8)}$$

Thus, for that case, the total solution is expressed as

$$\mathbf{x}^* = \left[\frac{h}{r+h}, \frac{h}{c}, \frac{h}{c}, \ldots, \frac{r}{r+h} - \frac{S^* h}{c}, 0, \ldots\right]. \qquad \textbf{(C.9)}$$

Cases in which c_i is not the same among species follow a similar pattern. To test local stability, we constructed the Jacobian, **J**, for this system:

$$\begin{bmatrix} -(h + c\sum_{i=1}^{S} x_i) - r & -cx_0 - r & -cx_0 - r & \cdots & -cx_0 - r \\ 0 & c\sum_{j=i}^{S} x_i & -(h + c\sum_{j=i+1}^{S} x_j) + cx_{i-1} & c(x_{i-1} - x_i) & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & cx_S & -h + cx_{s-1} \end{bmatrix} \qquad \textbf{(C.10)}$$

We substituted the solutions from Eq. C.9 for systems with randomly generated parameter values for S, r, c over harvest rates ranging from zero to 10. The eigenvalues, calculated with MATLAB, were always negative for these solutions, while other solutions (specifically, that for S^* one species lower) were unstable.