Appendix B - Proof of Eqs. A1.

Let χ_σ and χ_C be the characteristic polynomials of matrix $\mathbf{M}_\sigma$ and $\mathbf{C}$ respectively. Because $\mathbf{M}_\sigma = e\mathbf{C} + \sigma\mathbf{P}'\mathbf{C}$, the explicit computation of $\det(\mathbf{M}_\sigma - X\mathbf{I})$ gives $\chi_\sigma(X) = e^n\left(\chi_C(X/e) - \sigma Q_\sigma(X)\right)$, where Q_σ is a polynomial of degree $n-1$ which coefficients depend on σ. As λ_{M_σ} is an eigenvalue of $\mathbf{M}_\sigma$, it verifies $\chi_\sigma(\lambda_{M_\sigma}) = 0$, i.e., $\chi_C(\lambda_{M_\sigma}/e) = \sigma Q_\sigma(\lambda_{M_\sigma})$.

Assuming $\mathbf{C}$ is primitive and irreducible, the dominant eigenvalue λ_C of $\mathbf{C}$ is a single root of χ_C. There exists an open neighbourhood V of λ_C such that χ_C is continuously differentiable with nonzero derivative on V. Following the inverse function theorem, χ_C restricted to V is invertible, and its invert function, denoted χ_C^{-1}, is continuously differentiable. We then have $\lambda_{M_\sigma} = e\chi_C^{-1}\left(\sigma Q_\sigma(\lambda_{M_\sigma})\right)$.

$Q_\sigma(\lambda_{M_\sigma})$ is bounded for $\sigma \in V$, so a Taylor-Young expansion gives:

$$\begin{aligned}\lambda_{M_\sigma} &= e\chi_C^{-1}(0) + e\frac{1}{\chi_C' \circ \chi_C^{-1}(0)}\sigma Q_\sigma(\lambda_{M_\sigma}) + O(\sigma^2) \\ &= e\lambda_C + e\frac{1}{\chi_C'(\lambda_C)}\sigma Q_\sigma(\lambda_{M_\sigma}) + O(\sigma^2)\end{aligned}.$$

As λ_C is a single root of χ_C, $\chi_C'(\lambda_C) \neq 0$, and we conclude that $\lambda_{M_\sigma} = e\lambda_C + O(\sigma)$.

Let us denote $\mathbf{A}_\sigma = \mathbf{M}_\sigma - \lambda_{M_\sigma}\mathbf{I}$, and $\widetilde{\mathbf{A}}_\sigma$ its adjugate matrix. $\mathbf{N}_{\mathbf{M}_\sigma}$ is a positive vector with norm 1 which verifies $\mathbf{A}_\sigma\mathbf{N}_{\mathbf{M}_\sigma} = \mathbf{0}$. When σ is small enough, the order of multiplicity of λ_{M_σ} is 1 and the rank of $\mathbf{A}_\sigma$ is $n-1$. Then $\widetilde{\mathbf{A}}_\sigma$ has rank 1 and verifies $\mathbf{A}_\sigma\widetilde{\mathbf{A}}_\sigma = \mathbf{0}$, which implies that $\mathbf{N}_{\mathbf{M}_\sigma}$ is the unique nonnegative vector of norm 1 which is the generator of the columns of $\widetilde{\mathbf{A}}_\sigma$. Elements of $\widetilde{\mathbf{A}}_\sigma$ are polynomials of the elements of $\mathbf{A}_\sigma$. We deduce that $\mathbf{N}_{\mathbf{M}_\sigma}$ tends continuously toward the positive vector $\mathbf{N}_\mathbf{C}$ and verifies $\mathbf{N}_{\mathbf{M}_\sigma} = \mathbf{N}_\mathbf{C} + O(\sigma)$.