Appendix B: Mathematical proofs of equations (2) and (3)

Fix time t at t^*. Re-write eq. (1) as

$$\mathbf{x}_t = \mathbf{a} + \mathbf{z}t^* + \mathbf{C}\boldsymbol{\mu}_\mathbf{u} + \mathbf{B}\mathbf{x}_{t-1} + \mathbf{C}\mathbf{u}_t - \mathbf{C}\boldsymbol{\mu}_\mathbf{u} + \mathbf{e}_t \qquad t = 2,3,....$$

Set the constant $\mathbf{a} + \mathbf{z}t^* + \mathbf{C}\boldsymbol{\mu}_\mathbf{u}$ equal to $\tilde{\mathbf{a}}$, and set the random sum $\mathbf{C}\mathbf{u}_t - \mathbf{C}\boldsymbol{\mu}_\mathbf{u} + \mathbf{e}_t$ equal to $\tilde{\mathbf{e}}$. Note that the expectation of $\tilde{\mathbf{e}}$ is $\mathrm{E}\left[\mathbf{C}\mathbf{u}_t - \mathbf{C}\boldsymbol{\mu}_\mathbf{u} + \mathbf{e}_t\right] = \mathbf{C}\boldsymbol{\mu}_\mathbf{u} - \mathbf{C}\boldsymbol{\mu}_\mathbf{u} = \mathbf{0}$, and the variance of $\tilde{\mathbf{e}}$ is $\mathrm{Var}\left[\mathbf{C}\mathbf{u}_t - \mathbf{C}\boldsymbol{\mu}_\mathbf{u} + \mathbf{e}_t\right] = \mathbf{C}\boldsymbol{\Sigma}_\mathbf{u}\mathbf{C}^T + \boldsymbol{\Sigma}_\mathbf{e}$ (recall that $\mathbf{u}_t$ and $\mathbf{e}_t$ are assumed independent). Thus, eq. (1) can be re-written as

$$\mathbf{x}_t = \tilde{\mathbf{a}} + \mathbf{B}\mathbf{x}_{t-1} + \tilde{\mathbf{e}}_t \qquad t = 2,3,....$$

which is the multivariate AR(1) model from Ives et al. (2003) (their eq. 10). Thus, the mean and variance of the stationary distribution follow immediately as

$$\begin{aligned}\boldsymbol{\mu}_\mathbf{x} &= \left(\mathbf{I} - \mathbf{B}\right)^{-1}\tilde{\mathbf{a}} \\ &= \left(\mathbf{I} - \mathbf{B}\right)^{-1}\left(\mathbf{a} + \mathbf{C}\boldsymbol{\mu}_\mathbf{u} + \mathbf{z}t^*\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Vec}\left(\boldsymbol{\Sigma}_\mathbf{x}\right) &= \left(\mathbf{I} - \mathbf{B}\otimes\mathbf{B}\right)^{-1}\mathrm{Vec}\left(\mathrm{Var}\left(\tilde{\mathbf{e}}\right)\right) \\ &= \left(\mathbf{I} - \mathbf{B}\otimes\mathbf{B}\right)^{-1}\mathrm{Vec}\left(\mathbf{C}\boldsymbol{\Sigma}_\mathbf{u}\mathbf{C}^T + \boldsymbol{\Sigma}_\mathbf{e}\right).\end{aligned}$$