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A Periodogram of the OUSS model

Let $z_1, .., z_M$ be a real time series generated by a stationary OUSS model and measured at time intervals δ. Without loss of generality, we shall assume that the process mean is zero. Our starting point is the following definition of the periodogram

$$\widehat{s}(mf_o) = T \left| \frac{1}{M} \sum_{k=1}^{M} z_n e^{-2\pi i m(k-1)/M} \right|^2, \tag{A.1}$$

where $f_o = 1/(M\delta)$ and $m = 0, .., M/2$. We are interested in the expected value $s = \mathbb{E}\{\widehat{s}\}$, asymptotically for large M. Taking the expectation of Eq. (A.1) yields

$$s(mf_o) = \frac{T}{M^2} \sum_{k,l} \mathbb{E}\{z_k z_l\} e^{2\pi i m(k-l)/M}. \tag{A.2}$$

Recall that $z_k = x_k + y_k$, where x_k are generated by an OU process and y_k are centralized and uncorrelated with variance ε^2. The covariance, $\mathbb{E}\{z_k z_l\}$ between any two values z_k and z_l is given by $\sigma^2 e^{-\lambda\delta|k-l|} + \delta_{kl}\varepsilon^2$, where δ_{kl} denotes the Kronecker delta (Gillespie 1992). Hence Eq. (A.2) can be written as

$$\begin{aligned} s(mf_o) = \frac{T}{M^2} \sum_{k,l} &\left[\sigma^2 \rho^{|k-l|} + \varepsilon^2 \delta_{kl}\right] \\ &\times e^{2\pi i m(k-l)/M}, \end{aligned} \tag{A.3}$$

where $\rho = e^{-\lambda\delta}$. After a few algebraic manipulations one can write Eq. (A.3) as

$$\begin{aligned} s(f) = &\frac{s_o T(1-\rho)}{M^2\delta(1+\rho)} \left[M \frac{1-|r|^2}{|1-r|^2} \right. \\ &\left. - 2\Re\left\{ \frac{r(1-r^M)}{(1-r)^2} \right\} \right] + \delta\varepsilon^2, \end{aligned} \tag{A.4}$$

where we abbreviated

$$s_o = \frac{\delta\sigma^2(1+\rho)}{(1-\rho)}, \; r = \rho e^{2\pi i f\delta}, \; f = mf_o. \tag{A.5}$$

Note that s_o is the expected periodogram power at zero frequency in the limit $M \to \infty$, introduced in the main article. In that limit, Eq. (A.4) can be simplified and one obtains the asymptotic expression (Box et al. 2013, Eq. (3.2.15))

$$s(f) \sim \frac{\delta\sigma^2(1-\rho^2)}{1+\rho^2-2\rho\cos(2\pi f\delta)} + \delta\varepsilon^2. \tag{A.6}$$

If additionally $\lambda\delta \ll 1$ and $f\delta \ll 1$, and assuming $\varepsilon^2/\sigma^2 \in \mathcal{O}(1)$, one retrieves the power spectrum of the classical OU process,

$$s(f) \sim \frac{2\lambda\sigma^2}{(2\pi f)^2+\lambda^2} = S(f). \tag{A.7}$$

LITERATURE CITED

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