Appendix A – Reparameterizing the spatial and nonspatial Gompertz models as autoregressive models

Nonspatial Gompertz model

The nonspatial Gompertz model can be reparameterized as an autoregressive model:

$$\log(\boldsymbol{n}) = \frac{\omega}{1-\rho} + \phi\rho^{(\boldsymbol{t}\text{-}1)} + \boldsymbol{\psi} \tag{A.1}$$

where $\boldsymbol{t}$ is the vector of years $<1, 2, ..., T>$ and T is the maximum year, $\boldsymbol{n}$ is the vector of population abundances $<n_1, n_2, ..., n_T>$, and $\boldsymbol{\psi}$ is the vector of process variability $<\psi_1, \psi_2, \ldots, \psi_T>$:

$$\boldsymbol{\psi} \sim MVN(0, \boldsymbol{\Sigma}_\psi) \tag{A.2}$$

where $\boldsymbol{\Sigma}_\psi$ is temporal covariation, calculated as a function of process variance σ_ψ^2 and the autoregressive coefficient ρ (Clark 2007):

$$\boldsymbol{\Sigma}_\psi = \frac{\sigma_\psi^2}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & . & \rho^{T-1} \\ \rho & 1 & \rho & . & \rho^{T-2} \\ \rho^2 & \rho & 1 & . & \rho^{T-3} \\ . & . & . & . & . \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & . & 1 \end{bmatrix} \tag{A.3}$$

and where ψ_t in the autoregressive parameterization (Eq. 3a) equals $\varepsilon_t/sqrt(1\text{-}\rho^2)$ from the recursive parameterization (Eq. 1a).

Spatial Gompertz model

Similarly, the spatial Gompertz model can be re-parameterized as a vector autoregressive model:

$$\log(\mathbf{D}) = \frac{\alpha + \boldsymbol{\Omega}}{1-\rho} + \phi\rho^{(t-1)} + \boldsymbol{\Psi} \tag{A.4}$$

where $\boldsymbol{\Omega}$ is as defined previously, and $\boldsymbol{\Psi}$ now represents spatiotemporal error (i.e., the autoregressive structure for the vector autoregressive model):

$$\mathbf{\Psi} \sim MVN(0, \mathbf{\Sigma}_{\mathrm{E}} \otimes \mathbf{\Sigma}_{\psi}) \quad \text{(A.5)}$$

where $\mathbf{\Sigma}_{\mathrm{E}}$ and $\mathbf{\Sigma}_{\psi}$ are as defined previously. Here, $\mathbf{\Sigma}_{\mathrm{E}} \otimes \mathbf{\Sigma}_{\psi}$ is a Kronecker product which results in a sparse-matrix representation of the precision (inverse covariance) matrix of $\mathbf{\Psi}$ (which is advantageous from a computational perspective).

LITERATURE CITED

Clark, J. S. 2007. Models for Ecological Data: An Introduction. Princeton University Press, Princeton, New Jersey, USA.