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Lou Jost. 2007. Partitioning diversity into independent alpha and beta components. Ecology 88:2427–2439.

Appendix A. Proofs 1–4.

Proof 1: The gamma component H(${}^q\lambda_\gamma$) is by definition the diversity measure applied to the pooled communities, so that

$$ {}^q\lambda_\gamma = \sum_{i=1}^{S} [(w_1 p_{i1} + w_2 p_{i2} + \ldots)^q]. \tag{A.1} $$

When all communities are completely distinct (no shared species), this reduces to

$$ {}^q\lambda_{\gamma\ \mathrm{dist}} = \sum_{j=1}^{N} w_j^q ({}^q\lambda_j) \tag{A.2} $$

where ${}^q\lambda_j$ is $\sum_{i=1}^{S} p_{ij}^q$ for the jth community.

If ${}^qD_\alpha \cdot {}^qD_\beta = {}^qD_\gamma$ then

$$ {}^q\lambda_\alpha \cdot {}^q\lambda_\beta = {}^q\lambda_\gamma \tag{A.3} $$

by Eq. 2. From this and Eq. A.2 we can get an expression for ${}^q\lambda_\alpha$ when all communities are distinct:

$$ {}^q\lambda_{\alpha\ \mathrm{dist}} = [w_1^q ({}^q\lambda_1) + w_2^q ({}^q\lambda_2) + \ldots] / {}^q\lambda_{\beta\ \mathrm{dist}}. \tag{A.4} $$

When all communities have identical values of ${}^q\lambda$, so that ${}^q\lambda_j = {}^q\lambda$ for all j, ${}^q\lambda_{\alpha\,\mathrm{dist}}$ must also equal ${}^q\lambda$ by the assumption that the alpha component is some kind of "average" of the diversities of the individual communities. Therefore for this case

$$ {}^q\lambda_{\alpha\,\mathrm{dist}} = {}^q\lambda = [w_1{}^q({}^q\lambda) + w_2{}^q({}^q\lambda) + \ldots] / {}^q\lambda_{\beta\,\mathrm{dist}} \tag{A.5}$$

and by factoring out ${}^q\lambda$, the value of ${}^q\lambda_{\beta\,\mathrm{dist}}$ is determined:

$$ {}^q\lambda_{\beta\,\mathrm{dist}} = [w_1{}^q + w_2{}^q + \ldots] \tag{A.6}$$

Equation A.6 can be substituted into Eq. A.4 to yield an expression for ${}^q\lambda_\alpha$:

$$ {}^q\lambda_\alpha = [w_1{}^q({}^q\lambda_1) + w_2{}^q({}^q\lambda_2) + \ldots] / [w_1{}^q + w_2{}^q + \ldots]. \tag{A.7}$$

Because ${}^q\lambda_\alpha$ is by assumption independent of the beta component (which measures the distinctness of the communities), this result cannot depend on the beta component and must therefore apply even when the communities are not distinct. The alpha component of any standard diversity index is therefore $H({}^q\lambda_\alpha)$:

$$ H_\alpha = H({}^q\lambda_\alpha) = H([w_1{}^q({}^q\lambda_1) + w_2{}^q({}^q\lambda_2) + \ldots] / [w_1{}^q + w_2{}^q + \ldots]). \tag{A.8}$$

and the true alpha diversity of order q is

$$ ({}^q\lambda_\alpha)^{1/(1-q)} = \{[w_1{}^q({}^q\lambda_1) + w_2{}^q({}^q\lambda_2) + \ldots] / [w_1{}^q + w_2{}^q + \ldots]\}^{1/(1-q)}. \tag{A.9}$$

Proof 2: For $q \neq 1$ the true alpha diversity for the case of equal weights is, from Eq. 13,

$$ {}^qD_\alpha = [(1/N)(\sum_{i=1}^{S} p_{i1}^q) + (1/N)(\sum_{i=1}^{S} p_{i2}^q) + \ldots + (1/N)(\sum_{i=1}^{S} p_{iN}^q)]^{1/(1-q)}. \tag{A.10}$$

For $q > 1$ this alpha diversity will be less than or equal to gamma diversity if

$$ \sum_{i=1}^{S} (1/N)(p_{i1})^q + (1/N)(p_{i2})^q + \ldots + (1/N)(p_{iN})^q $$

$$ \geq \sum_{i=1}^{S} [(1/N)p_{i1} + (1/N)p_{i2} + \ldots + (1/N)p_{iN}]^q. \tag{A.11}$$

For q <1 this alpha diversity will be less than or equal to gamma diversity if

$$\sum_{i=l}^{S} (1/N)(p_{i1})^q + (1/N)\,(p_{i2})^q + \ldots + (1/N)(p_{iN})^q$$

$$\leq \sum_{i=l}^{S} [(1/N)p_{i1} + (1/N)p_{i2} + \ldots + (1/N)p_{iN}]^q.$$

These inequalities are necessarily true by Jensen's inequality (Renyi 1970). The case for q = 1 is also necessarily true, by the concavity of Shannon entropy (Lande 1996).

Proof 3: Consider for example Community A, with statistical weight 0.95, containing three species with frequencies {0.9, 0.05, 0.05}, and Community B, with weight 0.05, containing the same species with frequencies {0.8, 0.1, 0.1}. Whenever q is between 0.01and 0.9, this system violates Lande's condition. For example, when q = 0.5 the alpha diversity of this system is 2.10 effective species but the gamma diversity is only 2.08 effective species. An example of the violation of Lande's condition for all indices with q > 1.07 (thus including all Simpson indices, the Renyi entropy of order 1.07 or greater, and the Hurlbert-Smith-Grassle index with m = 2) is Community A with weight 0.995, containing three species with frequencies {0.7, 0.16, 0.14}, and Community B with weight 0.005, containing the same species with frequencies {0.88, 0.09, 0.03}. When q = 2 the alpha diversity of this system is 1.868 effective species but the gamma diversity is only 1.865 effective species.

Proof 4: If we have three equally diverse, equally large communities, the two that are most similar to each other in composition must have a lower pooled diversity than the two that are least similar to each other. Otherwise contradictions will arise; for example, the similarity measure could lead to the conclusion that A is most similar to B, but the diversity measure could lead to the conclusion that A is most similar to C.

Let A, B, C be large samples of equal sizes and equal diversities; let A+B represent the region consisting of A plus B, and let A+C represent the region consisting of A and C. For a given measure of similarity or overlap to be logically consistent with a given measure of diversity, the following condition must hold between them:

$$[\text{Similarity (A, B)} < \text{Similarity (A, C)}] \text{ if and only if } [\text{Diversity (A+B)} > \text{Diversity (A+C)}]. \tag{A.12}$$

This can be written:

$$[\text{Similarity (A, B)} < \text{Similarity (A, C)}] \text{ iff } [{}^qD_\gamma(A+B) > {}^qD_\gamma(A+C)]. \tag{A.13}$$

Using Eq. 9 the right-hand side can be written in terms of alpha and beta diversity as

$${}^qD_\alpha(A+B)\,{}^qD_\beta(A+B) > {}^qD_\alpha(A+C)\,{}^qD_\beta(A+C) \tag{A.14}$$

Since the diversities of A, B, and C are assumed to be equal, it follows that

$${}^qD_\alpha(A+B) = {}^qD_\alpha(A+C) = {}^qD(A) \tag{A.15}$$

and so the alpha diversities in Eq. A.14 can be divided out; therefore the consistency condition can be rewritten

$$[\text{Similarity (A, B)} < \text{Similarity (A, C)}] \text{ iff } [{}^qD_\beta(A+B) > {}^qD_\beta(A+C)]. \tag{A.16}$$

This condition can only be satisfied if the similarity indices are monotonically decreasing functions of the beta diversity; the species frequencies cannot enter into the calculation of the similarity index except through the beta diversity.

LITERATURE CITED

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