Appendix B: Methods for constructing projection matrices and computing stochastic elasticities

This appendix provides additional information on the analysis we performed. We first describe how we parameterized the growth and reversion rates for the perennial plant species in our analysis, for which life history stages were based on size. Then we describe how we calculated the elasticities to the means and standard deviations of all vital rates.

Parameterization of projection matrices for plants with size-structured populations

For the plant species in our analysis, most life-history stages were based on size rather than age. However, the number of size classes differed among species, due to choices made by the researchers who designed the original demographic studies. Therefore, the number of size classes in which an individual in a given size class this year might be found next year (and hence the number of vital rates representing different growth possibilities) differed among species. To make the vital rates more comparable across all plant species, in addition to size-class-specific survival probabilities and reproductive rates, we defined two additional overarching vital rates for each size class, one representing the overall probability of growing larger and the other representing the overall probability of reverting to smaller size classes. Other vital rates described the magnitudes of growth or reversion. For example, for a species with 6 above-ground size classes (seedlings as class 1 and 5 size-based classes) and no seed bank, the projection matrix would be parameterized as follows:

\[
A = \begin{bmatrix}
0 & s_6f_2 & s_6f_3 & s_6f_4 & s_6f_5 & s_6f_6 \\
1 & s_2(1-g_2) & s_3(1-g_3) & s_4(1-g_4) & s_5(1-g_5) & s_6(1-g_6) \\
0 & s_2g_2(1-h_{2,2}) & s_3g_3(1-h_{3,3}) & s_4g_4(1-h_{4,4}) & s_5g_5(1-h_{5,5}) & s_6g_6(1-h_{6,6}) \\
0 & s_2g_2h_{2,2}(1-h_{2,2}) & s_3g_3h_{3,3}(1-h_{3,3}) & s_4g_4h_{4,4}(1-h_{4,4}) & s_5g_5h_{5,5}(1-h_{5,5}) & s_6g_6h_{6,6}(1-h_{6,6}) \\
0 & s_2g_2h_{2,2}h_{2,2}(1-h_{2,2}) & s_3g_3h_{3,3}h_{3,3}(1-h_{3,3}) & s_4g_4h_{4,4}h_{4,4}(1-h_{4,4}) & s_5g_5h_{5,5}h_{5,5}(1-h_{5,5}) & s_6g_6h_{6,6}h_{6,6}(1-h_{6,6}) \\
0 & s_2g_2h_{2,2}h_{2,2}h_{2,2}(1-h_{2,2}) & s_3g_3h_{3,3}h_{3,3}h_{3,3}(1-h_{3,3}) & s_4g_4h_{4,4}h_{4,4}h_{4,4}(1-h_{4,4}) & s_5g_5h_{5,5}h_{5,5}h_{5,5}(1-h_{5,5}) & s_6g_6h_{6,6}h_{6,6}h_{6,6}(1-h_{6,6}) \\
\end{bmatrix}
\]

where \( f_i \) is the number of seeds produced by a plant in size class \( i \), \( s_i \) is the probability that a seed survives and germinates to become a seedling at the next census, \( s_i \) is the probability that...
an individual initially in size class $i$ survives one year, $g_i$ is the probability that, conditional on surviving, an individual initially in size class $i$ grows to a larger size class, $r_i$ is the probability that, conditional on surviving and not growing, an individual initially in size class $i$ reverts to a smaller size class, $h_{j,i}$ is the probability that, conditional on surviving and growing $j$ or more size classes, an individual initially in size class $i$ grows more than $j$ size classes, and $k_{j,i}$ is the probability that, conditional on surviving and growing $j$ or more size classes, an individual initially in size class $i$ reverts more than $j$ size classes. Note that the sum of the elements in rows 2 through 6 in column $i$ equals $s_i$ (as it should). Also note that $g_i$ is the overall probability of growing for a (surviving) individual in size class $i$, regardless of whether the individual grows 1, 2, etc. size classes, and similarly that $r_i$ is the overall probability of reverting given that an individual survives but doesn’t grow. The probability of an individual remaining in size class $i$ (conditional on not growing) therefore depends on both $g_i$ and $r_i$. The $g_i$’s and $r_i$’s are estimated with larger sample sizes than are the $h_{j,i}$’s and the $k_{j,i}$’s, and therefore we can put more confidence in the former. In Fig. B1, we report the elasticities of the $g_i$’s and $r_i$’s. Although we do not report those for the $h_{j,i}$’s and the $k_{j,i}$’s, we do include their elasticities in the total elasticity in Figs. 1, 2.

Computation and comparison of elasticities

We first used the vital rate estimates to construct projection matrices for each environmental state. We then drew a sequence of environmental states for $T=10^5$ years. If there was no evidence of or information about environmental autocorrelation, we assumed the environmental states were equally probable and IID across years. If, however, environmental autocorrelation was strongly indicated in the data, we drew the sequence of environments using the Markov environmental state transition matrix. The sequence of environments specifies a sequence of annual projection matrices, $A$, which we used to generate a sequence of population structures, $u$, forward in time and a sequence of reproductive value vectors, $v$, backward in time using the recursions

$$u(t+1) = A(t)u(t)/\lambda(t) \quad \text{and} \quad v'(t-1) = v'(t)A(t)/\|v'(t)A(t)\|$$

where $\lambda(t) = \|A(t)u(t)\|$ and $\|\|$ denotes the sum of vector elements. We discarded the first and last 500 vectors to allow $u$ and $v$ to approach their stationary distributions. The sequences $A(t)$, $u(t)$, $v(t)$, and $\lambda(t)$ for $t = 1$ to $T$, were then used to compute the elasticities of $\lambda_i$, using the equation (Tuljapurkar et al. 2003, Haridas and Tuljapurkar 2005)

$$E_{\lambda_i} = \frac{1}{T} \sum_{t=1}^{T} \frac{v'(t)C_i(t)u(t)}{\lambda(t)v'(t)u(t+1)}$$

where $C_i(t)$ is a matrix of perturbations to $A(t)$ resulting from perturbations to the $i$th vital rate, $v_i$ (the mean of which, averaged across environmental states weighted by their frequencies, is
Specifically, if \( \mathbf{C}_i(t) = \bar{v}_i \frac{\partial \mathbf{A}}{\partial v_i} \), then \( E_{\bar{v}_i} \equiv E_{\bar{v}_i}^\mu \) is the elasticity of \( \lambda_s \) to a change in the mean of vital rate \( i \), whereas if \( \mathbf{C}_i(t) = (v_i(t) - \bar{v}_i) \frac{\partial \mathbf{A}}{\partial v_i} \), then \( E_{v_i} \equiv E_{v_i}^\sigma \) is the elasticity of \( \lambda_s \) to a change in the standard deviation of vital rate \( i \) (where in both cases, the derivative is evaluated at the values of the vital rates in year \( t \)). Strictly speaking, the perturbation used to compute \( E_{v_i}^\sigma \) changes the variance and all higher central moments of \( v_i \) (Haridas and Tuljapurkar 2005), as well as the covariances between vital rates, but for simplicity we refer to \( E_{v_i}^\sigma \) as the elasticity to the standard deviation of \( v_i \).

**LITERATURE CITED**
